THE BONDI-SACHS FORMALISM

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NULL HYPERSURFACES \( u = \text{const} \)

Normal co-vector \( \partial_\alpha u \) is null \( g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0 \)

Normal vector \( k^\alpha = -g^{\alpha\beta} \partial_\beta u \) is also tangent vector
\[
k^\alpha \partial_\alpha u = -g^{\alpha\beta} \partial_\beta u \partial_\alpha u = 0
\]

Null rays \( x^\alpha(\lambda) \)
\[
\frac{\partial x^\alpha}{\partial \lambda} = k^\alpha
\]

Null rays are null geodesics
\[
k^\alpha \nabla_\alpha k_\beta = -k^\alpha \nabla_\alpha \nabla_\beta u = -k^\alpha \nabla_\beta \nabla_\alpha u = k^\alpha \nabla_\beta k_\alpha = \frac{1}{2} \nabla_\beta (k^\alpha k_\alpha) = 0
\]

The local null cone

If \( T^\alpha \) is timelike or null then \( T^\alpha k_\alpha \neq 0 \) unless \( T^\alpha = k^\alpha \)

If \( V^\alpha \) is tangent to \( u = \text{const} \) so that \( V^\alpha \partial_\alpha u = 0 \) then \( V^\alpha \) is spacelike unless \( V^\alpha = k^\alpha \)
MINKOWSKI SPACE

Cartesian inertial coordinates \( x^\alpha = (t, x^i) = (t, x, y, z) \)

Minkowski metric \( \eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1) \)

**Future nullcone:** \( u = t - r \), \( r^2 = \delta_{ij} x^i x^j \)

**Cartesian components of normal:** \( k_\alpha = -\partial_\alpha u = (-1, x_i/r) \)

**Tangent vector** \( k^\alpha = \eta^{\alpha\beta} k_\beta = (1, x^i/r) \)

\( k^\alpha k_\alpha = 0 \)

**Null coordinates** \( x^\alpha = (u, r, x^A) \), \( x^A = (\theta, \phi) \)

\( ds^2 = -dt^2 + dr^2 + r^2 q_{AB} dx^A dx^B = -du^2 - 2dudr + r^2 q_{AB} dx^A dx^B \)

**Unit sphere metric** \( q_{AB} dx^A dx^B = d\theta^2 + \sin^2 \theta d\phi^2 \)

**In null coordinates** \( k^\alpha \partial_\alpha u = k^\alpha \partial_\alpha x^A = 0 \), \( k^\alpha \partial_\alpha = \partial_r \)
Curved space Bondi–Sachs coordinates

Null coordinates: \( x^\alpha = (u, r, x^A) \)

\( u = \text{const} \): Family of outgoing null hypersurfaces

\( r \): Areal radial coordinate along null rays

\( x^A = (\theta, \phi) \): Angular coordinates constant along null rays

Future pointing tangent vector to null rays \( k^\alpha = -g^{\alpha\beta} \partial_\beta u \)

Coordinate Conditions on Metric Components

\[ 0 = g^{\alpha\beta}(\partial_\alpha u)\partial_\beta u \Rightarrow g^{uu} = 0 \]

\[ 0 = k^\alpha \partial_\alpha x^A = -g^{\alpha\beta}(\partial_\alpha u)\partial_\beta x^A = -g^{\alpha\beta}(\partial_\alpha u)\delta^A_\beta \Rightarrow g^{uA} = 0 \]

Areal coordinate \( r \) \( \Rightarrow \) \( \det[g_{AB}] = r^4 \det[q_{AB}] = r^4 \sin^2 \theta \)

\( \Rightarrow g_{AB} = r^2 h_{AB} \)

\( \det[h_{AB}] = \det[q_{AB}] = q(x^A) \)

Covariant Bondi-Sachs metric

\[ \delta^\alpha_\gamma = g^{\alpha\beta} g_{\beta\gamma} \]

\[ \delta^u_r = 0 = g^{ur} g_{rr} \Rightarrow g_{rr} = 0 \]

\[ \delta^u_A = 0 = g^{ur} g_{rA} \Rightarrow g_{rA} = 0 \]

\[ g_{\alpha\beta} dx^\alpha dx^\beta = -\frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} du dr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du) \]

\(\det[h_{AB}] = q \) so conformal 2-metric \( h_{AB} \) has only two degrees of freedom

Non-zero contravariant components

\[ g^{ur} = -e^{-2\beta} \ , \ \ g^{rr} = \frac{V}{r} e^{-2\beta} \ , \ \ g^{rA} = -U^A e^{-2\beta} \ , \ \ g^{AB} = \frac{1}{r^2} h^{AB} \]
The Electromagnetic Analogue

Minkowski metric in outgoing null coordinates \((u, r, x^A)\)

\[
\eta_{\alpha\beta} dx^\alpha dx^\beta = -du^2 - 2drdu + r^2 q_{AB} dx^A dx^B
\]

Assume sources of electromagnetic field are inside a world-tube \(\Gamma\), with spherical cross-sections of radius \(r = R\). The outgoing null cones \(N_u\) from \(r = 0\) intersect \(\Gamma\) in spheres \(S_u\) coordinatized by \(x^A\)

The Maxwell field \(F_{\alpha\beta}\) is represented by vector potential \(A_\alpha\),

\[
F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \text{ with gauge freedom } A_\alpha \to A_\alpha + \nabla_\alpha \chi
\]

Choice \(\chi(u, r, x^A) = -\int_{0}^{r} A_r dr'\) leads to null gauge \(A_r = 0\), which is analogue of Bondi-Sachs coordinate conditions \(g_{rr} = g_{rA} = 0\)

Remaining freedom \(\chi(u, x^A)\) used to set \(\lim_{r \to \infty} A_\alpha(u, r, x^A) = 0\), i.e. \(A_u|_{I^+} = 0\) where \(I^+\) is future null infinity

Remnant freedom \(A_B \to A_B + \nabla_B \chi(x^C)\) is analogue of Bondi-Metzner-Sachs supertranslation freedom

E-mode gauge shift on radiation field \(A_B|_{I^+}\)
Bondi-Sachs Version of Maxwell Equations

Source-free Maxwell equations $M^\beta := \nabla_\alpha F^{\alpha\beta} = 0$

\[ F^{\alpha\beta} = -F^{\beta\alpha} \Rightarrow \]

**IDENTITY** \[ 0 \equiv \nabla_\beta M^\beta = \nabla_\beta \nabla_\alpha F^{\alpha\beta} = 0 \]

\[ \nabla_\beta M^\beta = \frac{1}{\sqrt{-g}} \partial_\beta (\sqrt{-g} M^\beta) = \partial_u M^u + \frac{1}{r^2} \partial_r (r^2 M^r) + \frac{1}{\sqrt{q}} \partial_A (\sqrt{q} M^A) \]

**Strategy**

Designate $M^u = 0$ and $M^A = 0$ as Main Equations

Designate $M^r = 0$ as Supplementary Condition

If Main Equations satisfied then IDENTITY implies

\[ 0 = \partial_r (r^2 M^r) \]

so Supplementary Condition is satisfied if it is satisfied at any specified value of $r$, e.g. on $\Gamma$ or at $\mathcal{I}^+$

Main Equations separate into

**Hypersurface Equation** $M^u = 0$

\[ \partial_r (r^2 \partial_r A_u) = \partial_r (\bar{\partial}_B A^B) \]

**Evolution Equation** $M^A = 0$

\[ \partial_r \partial_u A_B = \frac{1}{2} \partial_r^2 A_B - \frac{1}{2} r^2 \bar{\partial}^C (\bar{\partial}_B A_C - \bar{\partial}_C A_B) + \frac{1}{2} \partial_r \bar{\partial}_B A_u \]

$\bar{\partial}_A$ is covariant derivative with respect to $q_{AB}$, $\bar{\partial}_A = q^{AB} \bar{\partial}_B$
Supplementary Equation at $\mathcal{I}^+$

$M^r = 0 : \quad \partial_u (r^2 \partial_r A_u) = \tilde{\partial}^B (\partial_r A_B - \partial_u A_B + \tilde{\partial}_B A_u)$

Integration over sphere at $\mathcal{I}^+$

$\Rightarrow$ Conservation Law $\quad \partial_u \int r^2 \partial_r A_u \sin \theta d\theta d\phi \mid_{\mathcal{I}^+} = 0$

**INTERPRETATION**

Hypersurface Equation $\quad \partial_r (r^2 \partial_r A_u) = \partial_r (\tilde{\partial} B A^B)$

Radial integral $\Rightarrow \quad r^2 \partial_r A_u = Q(u, x^A) + \tilde{\partial} B A^B$

Null gauge $A_r = 0$ so E-field component $E_r = F_{ru} = \partial_r A_u$

Charge Aspect $Q(u, x^A)$ is function of integration

Integral over sphere at $\mathcal{I}^+$

$Q(u) := \frac{1}{4\pi} \int r^2 \partial_r A_u \sin \theta d\theta d\phi \mid_{\mathcal{I}^+} = \frac{1}{4\pi} \oint Q(u, x^A) \sin \theta d\theta d\phi$

Supplementary Condition at $\mathcal{I}^+$ $\Rightarrow$

Charge Conservation $\quad \frac{dQ(u)}{du} = 0$
Worldtube-Null Cone Evolution Algorithm

Initial data $A_B|_{N_{u_0}}$

Initial wordtube (boundary) data $\partial_r A_u|_{S_{u_0}}$

Worldtube data $\partial_u A_B|_{\Gamma}$ (Dynamical degrees of freedom)

$A_u|_{\Gamma}$ (Gauge)

Sequential radial integration scheme

Hypersurface Eq: $\partial_r (r^2 \partial_r A_u) = \partial_r (\bar{\delta}_B A^B)$

$\Rightarrow A_u|_{N_{u_0}}$

Evolution Eq: $\partial_r \partial_u A_B = \frac{1}{2} \partial_r^2 A_B - \frac{1}{2} r^2 \bar{\delta}^C (\bar{\delta}_B A_C - \bar{\delta}_C A_B) + \frac{1}{2} \partial_r \bar{\delta}_B A_u$

$\Rightarrow \partial_u A_B|_{N_{u_0}}$

Finite difference approximation $\Rightarrow A_B|_{N_{u_0 + \Delta u}}$

Supplementary Condition $r^2 \partial_u \partial_r A_u = \bar{\delta}^B (\partial_r A_B - \partial_u A_B + \bar{\delta}_B A_u)$

$\Rightarrow \partial_u \partial_r A_u|_{S_{u_0}}$

Finite difference approximation $\Rightarrow \partial_r A_u|_{S_{u_0 + \Delta u}}$

Given $A_B|_{N_{u_0 + \Delta u}}$ and $\partial_r A_u|_{S_{u_0 + \Delta u}}$ iteration gives

finite difference approximation for $A_B|_{N_{u_0 + n\Delta u}}$ and $\partial_r A_u|_{S_{u_0 + n\Delta u}}$

Convergent finite difference scheme gives analytic solution
Bondi-Sachs Gravitational Evolution Algorithm

Null coordinates $x^\alpha = (u, r, x^A)$

$$g^{uu} = g^{uA} = 0, \quad g^{AB} = r^{-2}h^{AB}, \quad h^{AB}h_{BC} = \delta^A_C$$

$$\text{det}[h_{AB}] = q(x^C) \quad \Rightarrow \quad h^{AB}\partial_r h_{AB} = 0, \quad h^{AB}\partial_u h_{AB} = 0$$

Einstein equations: $E_{\alpha\beta} := G_{\alpha\beta} - 8\pi T_{\alpha\beta} = 0$

Bianchi identities $\nabla_\beta G^\beta_\alpha = 0$ and matter conservation $\nabla_\beta T^\beta_\alpha = 0$

$$\Rightarrow \quad 0 = \nabla_\beta E^\beta_\alpha = \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}E^\beta_\alpha) + \frac{1}{2}(\partial_\alpha g^{\beta\gamma})E_\beta\gamma$$

Analogous to electromagnetic case, designate

Main Equations: $E^\beta_r = 0, \quad E_{AB} - \frac{1}{2}g_{AB}g^{CD}E_{CD} = 0$

Bianchi-matter identities imply

$$\nabla_{\beta}E^{\beta}_r = \frac{1}{2}(\partial_{r}g^{\beta\gamma})E_{\beta\gamma} = \frac{1}{2}(\partial_{r}g^{BC})E_{BC} = \frac{1}{4}(\partial_{r}g^{BC})g_{BC}(g^{AD}E_{AD})$$

$$(\partial_{r}g^{BC})g_{BC} = -\frac{4}{r} \quad \Rightarrow \quad \text{Trivial Equation:} \quad g^{AD}E_{AD} = 0$$

Remaining components reduce to

$$\partial_{r}(\sqrt{-g}E^r_u) = 0, \quad \partial_{r}(\sqrt{-g}E^r_A) = 0$$

$$\Rightarrow \quad \text{Supplementary Conditions:} \quad E^r_u = E^r_A = 0$$

Satisfied if satisfied on worldtube $\Gamma$ or at $\mathcal{I}^+$

At $\mathcal{I}^+$ Supplementary Conditions give conservation laws for energy-momenum and angular momentum

$$r^2E^r_u|_{\mathcal{I}^+} = 0 \text{ gives rise to the famous Bondi mass loss equation}$$
Main Equations

Separate into Hypersurface Eqs: \( E^\alpha_r = 0 \)

Evolution Eqs: \( E_{AB} - \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0 \)

Bondi-Sachs metric

\[
g_{\alpha\beta} dx^\alpha dx^\beta = -\frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} dudr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du)
\]

Hypersurface Eqs are sequence of radial ODEs for metric variables \((\beta, U^A, V)\)

\[
\partial_r \beta = \frac{r}{16} h^{AC} h^{BD}(\partial_r h_{AB})(\partial_r h_{CD}) + 2\pi r T_{rr}
\]

\[
\partial_r \left[r^4 e^{-2\beta} h_{AB}(\partial_r U^B)\right] = 2r^4 \partial_r (\frac{1}{r^2} D_A \beta) - r^2 h^{EF} D_E (\partial_r h_{AF}) + 16\pi r^2 T_{rA}
\]

\[
2e^{-2\beta} (\partial_r V) = \mathcal{R} - 2[D_A D^A \beta + (D_A \beta)(D^A \beta)]
\]

\[
+ \frac{e^{-2\beta}}{r^2} D_A[\partial_r (r^4 U^A)] - \frac{1}{2} r^4 e^{-4\beta} h_{AB}(\partial_r U^A)(\partial_r U^B)
\]

\[
+ 8\pi [h^{AB} T_{AB} - r^2 T_{\alpha}]
\]

where \(D_A\) is covariant derivative and \(\mathcal{R}\) the Ricci scalar with respect to the conformal 2-metric \(h_{AB}\)

Vacuum Hypersurface Equations

Given null data \(h_{AB}|_{N_u}\) and worldtube data \((\beta, U^A, \partial_r U^A, V)|_{\Gamma}\) the Hypersurface Eqs \(\Rightarrow (\beta, U^A, \partial_r U^A, V)|_{N_u}\)
Evolution Equations

Evolution Eqs can be simplified by introducing a complex null dyad $m^a$ tangent to $N_u$, $m^\alpha \nabla_\alpha u = 0$, whose components span the angular directions so that $m^\alpha = (0, 0, m^A)$ with $h_{AB} m^A m^B = 0$

**Normalization** $m_A \bar{m}^A = 2$, $m_A = h_{AB} m^B$  
$\Rightarrow h^{AB} = \frac{1}{2}(m^A \bar{m}^B + m^B \bar{m}^A)$,

$\Rightarrow$ Determines $m^A$ up to phase freedom $m^A \rightarrow e^{i\eta} m^A$, which can be fixed by convention

**Evolution Eqs reduce to complex equation**  
$E_{AB} - \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0 \Rightarrow m^A m^B E_{AB} = 0$

**In terms of metric variables**

$$m^A m^B \{r \partial_r (r \partial_u h_{AB}) - \frac{1}{2} \partial_r (r V \partial_r h_{AB}) - 2 e^{\beta} D_A D_B e^\beta$$

$$+ h_{CA} D_B \partial_r (r^2 U^C) - \frac{1}{2} r^4 e^{-2\beta} h_{AC} h_{BD} (\partial_r U^C) \partial_r U^D$$

$$+ \frac{r^2}{2} (\partial_r h_{AB}) D_C U^C + r^2 U^C D_C \partial_r h_{AB}$$

$$- r^2 (\partial_r h_{AC}) h_{BE} (D^C U^E - D^E U^C) - 8\pi e^{2\beta} T_{AB} \} = 0$$

Radial ODE which determines the retarded time derivative of the two degrees of freedom in the conformal 2-metric $\partial_u h_{AB}$
Worldtube-Null-Cone Problem

Given null data $h_{AB}|_{N_u}$ and worldtube data $\partial_u h_{AB}|_{\Gamma}$ and $(\beta, U^A, \partial_r U^A, V)|_{\Gamma}$ which satisfy the supplementary conditions the Main Eqs determine a finite difference approximation for $h_{AB}|_{N_{u+\Delta u}}$

Cauchy-Characteristic Extraction

Worldtube data on $\Gamma$ obtained from numerical solution of Einstein’s equations carried out by Cauchy evolution of interior so that data satisfy the supplementary conditions

Given the initial-boundary data the hypersurface equations can be integrated numerically in sequential order and the evolution equation can be solved using a finite difference time-integrator

Implemented as stable, convergent evolution code which propagates the exterior solution to $I^+$

The resulting waveform can be unambiguously computed at $I^+$ without near field contamination
Asymptotic Behavior

Bondi-Sachs asymptotic solution based upon a $1/r$ expansion, with compact matter distribution

Bondi-Sachs metric approaches Minkowski metric at $I^+$

$$\Rightarrow \lim_{r \to \infty} \beta = \lim_{r \to \infty} U^A = 0, \quad \lim_{r \to \infty} \frac{V}{r} = 1, \quad \lim_{r \to \infty} h_{AB} = q_{AB}$$

INTEGRATION OF MAIN EQUATIONS REQUIRES

INITIAL DATA ON $N_0$

Conformal 2-metric $h_{AB} = q_{AB} + \frac{1}{r} c_{AB}(u_0, x^E) + \frac{1}{r^2} d_{AB}(u_0, x^E) + ...$

$$\det(h_{AB}) = q(x^C) \Rightarrow q^{AB} c_{AB} = 0, \quad q^{AB} d_{AB} = \frac{1}{2} c^{AB} c_{AB}$$

Mass aspect $M(u_0, x^A) = -\frac{1}{2}[V(u_0, r, x^C) - r]|_{I^+}$

Angular momentum aspect $L^A(u_0, x^C) = -\frac{1}{6} r^4 \partial_r U^A(u_0, r, x^C)|_{I^+}$

Schwarzschild data $h_{AB} = q_{AB}, \quad \beta = L^A = 0, \quad M = m \Rightarrow$

Eddington-Finklestein $ds^2 = -(1 - \frac{2m}{r})du^2 - 2dudr + r^2 q_{AB} dx^A dx^B$

RADIATION DATA ON $I^+$

News tensor $N_{AB} = \frac{1}{2} \partial_u c_{AB}(u, x^C)$

Strain tensor $c_{AB}(u, x^C) = r(h_{AB} - q_{AB})|_{I^+}$

Complex polarization dyad on unit sphere

$q^{AB} = \frac{1}{2}(q^A q^B + q^B q^A) \quad q^A = (1, i/\sin \theta) = m^A|_{I^+}$

Bondi news function $N = q^A q^B N_{AB} = N_\oplus + iN_\odot$

Radiation strain $\sigma = \frac{1}{2} q^A q^B c_{AB} = \sigma_\oplus + i\sigma_\odot$

Asymptotic shear of $N_u \quad \sigma = q^A q^B \nabla_A \nabla_B u|_{I^+}$
Integration of Main Equations

**Hypersurface Equations**

\[ \beta = -\frac{1}{32r^2} c^{AB} c_{AB} + O(r^{-3}) \]

\[ \partial_r \left[ r^4 e^{-2\beta} h_{AB}(\partial_r U^B) \right] = \bar{\sigma}^E c_{AE} + \frac{1}{r} S_A + O(r^{-2}) \]

where \( S_A = \bar{\sigma}^B (2 d_{AB} - q^{FG} c_{BG} c_{AF}) \)

Smoothness at \( I^+ \) \( \Rightarrow \) Require \( S_A = 0 \) to avoid \( r^{-4} \ln r \) term in \( \partial_r U^A \)

Smoothness on sphere: \( S_A = 0 \) \( \Rightarrow \) \( q^A q^B d_{AB} = 0 \)

so that \( d_{AB} = \frac{1}{2} q_{AB} q^{CD} d_{CD} \)

(See Scholarpedia article)

**Integration of smooth hypersurface data**

\[ U^A = -\frac{1}{2r^2} \bar{\sigma}^B c^{AB} + \frac{2}{r^3} L^A + O(r^{-4}) \]

\[ V = r - 2M(u_0, x^A) + O(r^{-1}) \]

Evolution Equation

Determines \( \partial_r \partial_u (r h_{AB}) \)

Leading order \( \Rightarrow \) \( q^A q^B \partial_u d_{AB} = 0 \)

Consistent with smoothness condition
Supplementary conditions at $\mathcal{I}^+$

$E^r_u$ supplementary equation

$$2\partial_u M = -\frac{1}{2}\partial_A \partial_B N^{AB} - \frac{1}{4} N_{AB} N^{AB}$$

Radiation data - News tensor $N_{AB}(u, x^A) \Rightarrow M(u, x^A)$

BondiMass $m(u) = \frac{1}{4\pi} \int M(u, \theta, \phi) \sin \theta d\theta d\phi$

Bondi mass loss formula $\frac{d}{du} m(u) = -\frac{1}{4\pi} \int |N|^2 \sin \theta d\theta d\phi$

$E^r_A$ supplementary equation

$$-3\partial_u L_A = \partial_A M - \frac{1}{4} \partial^E (\partial_E \partial^F c_{AF} - \partial_A \partial^F c_{EF})$$

$$+ \frac{1}{8} c_{EF} \partial_A N^{EF} - \frac{3}{8} N_{EF} \partial_A c^{EF} - N_{AB} \partial_E c^{BE} - c_{AB} \partial_E N^{BE}$$

$$\Rightarrow L(u, x^A)$$

One motivation for calling $L_A(u, x^A)$ the angular momentum aspect results from coupling to matter where $r^2 T^r_A$ is angular momentum flux of matter to $\mathcal{I}^+$

Asymptotic integration approach shows how initial data and radiation data determine solution as $1/r$ expansion about $\mathcal{I}^+$

However, this is not a well-posed evolution problem because the radiation data $N_{AB}(u, x^C)$, lies in the future of initial hypersurface at $u_0$

Assigning radiation data apriori is non-physical as opposed to determining $N$ by evolving an interior system, as in nullcone-worldtube problems

Nevertheless, approach led Bondi to the first clear understanding of mass loss due to gravitational radiation
Penrose compactification of $\mathcal{I}^+$

Geometric picture of asymptotic behavior

Metric of a conformally compactified spacetime $\hat{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ related to physical metric $g_{\alpha\beta}$ by conformal factor satisfying $\Omega = 0$ at $\mathcal{I}^+$

Distance to $\mathcal{I}^+$ is finite measured by $\hat{g}_{\alpha\beta}$

Asymptotic flatness requires $\mathcal{I}^+$ has topology $\mathbb{R} \times S^2$ and $\hat{\nabla}_\alpha \Omega|_{\mathcal{I}^+} \neq 0$, $\hat{\nabla}_\alpha$ is covariant derivative associated with $\hat{g}_{\alpha\beta}$

Einstein Equations

Conformal and physical Ricci tensors related by

$$\Omega^2 R_{\alpha\beta} = \Omega^2 \hat{R}_{\alpha\beta} + 2\Omega \hat{\nabla}_\alpha \hat{\nabla}_\beta \Omega + \hat{g}_{\alpha\beta} [\Omega \hat{\nabla}_\gamma \hat{\nabla}_\gamma \Omega - 3(\hat{\nabla}_\gamma \Omega) \hat{\nabla}_\gamma \Omega]$$

Trace of vacuum Einstein equations $R^\alpha_\alpha = 0$ at $\mathcal{I}^+$

$$0 = (\hat{\nabla}^\gamma \Omega) \hat{\nabla}_\gamma \Omega|_{\mathcal{I}^+} \Rightarrow \mathcal{I}^+ \text{ is null with tangent } n^\alpha = \hat{\nabla}^\alpha \Omega$$

Next order in $\Omega$ \Rightarrow \quad $$0 = [\hat{\nabla}_\alpha \hat{\nabla}_\beta \Omega - \frac{1}{4} \hat{g}_{\alpha\beta} \hat{\nabla}^\gamma \hat{\nabla}_\gamma \Omega]|_{\mathcal{I}^+}$$

Preferred Conformal Factors

Existence of a conformal transformation $\Omega \rightarrow \omega \Omega$ such that

$$\hat{\nabla}^\alpha \hat{\nabla}_\alpha \Omega \rightarrow \omega \hat{\nabla}^\alpha \hat{\nabla}_\alpha \Omega + \hat{\nabla}^\alpha \Omega \hat{\nabla}_\alpha \omega = 0 \quad \text{(ODE for } \omega)$$

$\Rightarrow$ Preferred conformal factors such that $\hat{\nabla}_\alpha \hat{\nabla}_\beta \Omega|_{\mathcal{I}^+} = 0$

$\mathcal{I}^+$ is shear and divergence free
Bondi-Sachs Compactification

Coordinates \( \hat{x}^\alpha = (u, \ell, x^A) \) of compactified space can be obtained from Bondi-Sachs coordinates of physical space by transformation \( \hat{x}^\alpha = (u, \ell, x^A) = (u, 1/r, x^A) \)

Inverse areal coordinate \( \ell = 1/r \) is also preferred conformal factor \( \Omega = \ell \) which leads to the conformal Bondi-Sachs metric

\[
\hat{g}_{ab} dx^a \hat{x}^b = \ell^3 V e^{2\beta} du^2 + 2e^{2\beta} dud\ell + h_{AB}(dx^A - U^A du)(dx^B - U^B du)
\]

Smoothness of \( \hat{g}_{ab} \) (at least \( C^3 \)) implies \( 1/r = \ell \) expansion at \( \mathcal{I}^+ \) which rules out troublesome \( \ln \) behavior and leads to peeling property \( \hat{C}_{\alpha\beta\gamma\delta} = O(\ell) \)

Einstein equations determine general leading coefficients of the conformal space metric

\[
\begin{align*}
    h_{AB} &= H_{AB}(u, x^C) + \ell c_{AB}(u, x^c) + O(\ell^2) \\
    \beta &= H(u, x^C) + O(\ell^2) \\
    U^A &= H^A(u, x^C) + 2\ell e^{2H} H^{AB} D_B H + O(\ell^2) \\
    \ell^2 V &= D_A H^A + \ell \left( \frac{1}{2} \mathcal{R} + D^A D_A e^{2H} \right) + O(\ell^2)
\end{align*}
\]

where \( \mathcal{R} \) is Ricci scalar and \( D_A \) the covariant derivative associated with \( H_{AB} \)

Leading coefficients \( H, H^A \) and \( H^{AB} \) do not correspond to an asymptotic inertial frame - Pure gauge
Construction of Bondi-Sachs Inertial Coordinates

General leading coefficients of the conformal space metric

\[ h_{AB} = H_{AB}(u, x^C) + \ell c_{AB}(u, x^c) + O(\ell^2) \quad \text{det}[h_{AB}] = q(x^C) \]

\[ \beta = H(u, x^C) + O(\ell^2) \]

\[ U^A = H^A(u, x^C) + 2\ell e^{2H} H^{AB} D_B H + O(\ell^2) \]

\[ \ell^2 V = D_A H^A + \ell(\frac{1}{2} \mathcal{R} + D^A D_A e^{2H}) + O(\ell^2) \]

\[ \hat{g}^{\alpha\beta}|_{I^+} = \begin{pmatrix} 0 & e^{-2H} & 0 \\ e^{-2H} & 0 & -H^A e^{-2H} \\ 0 & -H^A e^{-2H} & H^{AB} \end{pmatrix} \]

\((H, H^A, H^{AB})\) Gauge terms

Tangent to null geodesics on \(I^+\)

\[ n^\alpha|_{I^+} = \hat{g}^{ab}\nabla_b \ell|_{I^+} = (e^{-2H}, 0, -e^{-2H} H^A) \]

Inertial angular coordinates \(n^\alpha \partial_\alpha x^A|_{I^+} = 0 \Rightarrow H^A = 0\)

Inertia retarded time coordinate \(u\) - affine parameter

\[ n^\alpha \partial_\alpha u|_{I^+} = 1 \quad \Rightarrow \quad H = 0 \quad \Rightarrow \quad n^\alpha \partial_\alpha|_{I^+} = \partial_u \]

\(\ell\) is preferred conformal factor:

\[ \sqrt{-\det[\hat{g}_{\alpha\beta}]} = e^{2\beta} \sqrt{q(x^C)} \]

\[ \Rightarrow \hat{\nabla}_\alpha \hat{n}^\alpha|_{I^+} = \frac{1}{e^{2\beta} \sqrt{q}} \partial_\alpha(e^{2\beta} \sqrt{q} \hat{g}^{(\alpha)}|_{I^+} = \frac{1}{e^{2\beta} \sqrt{q}} \partial_u(e^{2\beta} \sqrt{q} e^{-2\beta})|_{I^+} = 0 \]

\[ \Rightarrow I^+ \text{ is divergence free and shear free} \Rightarrow \partial_u H_{AB} = 0 \]

Allows \(u\)-independent conformal transformation \(\ell \rightarrow \omega(x^C)\ell\)

such that \(H_{AB} \rightarrow q_{AB}\)

\(\Rightarrow\) Cross-sections of \(I^+\) have unit sphere geometry \(\Rightarrow \mathcal{R} = 2\)

Bondi inertial conformal frame \(H = H^A = 0, \quad H^{AB} = q^{AB}\)
Universal Structure of $\mathcal{I}^+$

Conformal metric in inertial Bondi-Sachs coordinates

$$\hat{g}_{\alpha\beta}d\hat{x}^\alpha d\hat{x}^\beta = \ell^2 g_{\alpha\beta}dx^\alpha dx^\beta \quad \hat{x}^\alpha = (u, \ell, x^A) \quad \ell = 1/r$$

$$\hat{g}_{\alpha\beta}d\hat{x}^\alpha d\hat{x}^\beta = \ell^3 V e^{2\beta} du^2 + 2e^{2\beta} dud\ell + h_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

$$h_{AB} = q_{AB} + \ell c_{AB}(u, x^C) + O(\ell^2)$$

$$\beta = O(\ell^2)$$

$$U^A = -\frac{1}{2}\ell^2 \delta_{BC}^{AB} + 2L^A \ell^3 + O(\ell^4)$$

$$\ell^3 V = \ell^2 - 2M \ell^3 + O(\ell^4)$$

$$\hat{g}^{\alpha\beta}|_{\mathcal{I}^+} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q^{AB} \end{pmatrix}$$

Shear-free property of $\mathcal{I}^+$ \Rightarrow \ell^{-1}\hat{\nabla}_\alpha \hat{\nabla}_\beta \ell$ has finite limit

News tensor $N_{\alpha\beta} = \ell^{-1}\hat{\nabla}_\alpha \hat{\nabla}_\beta \ell|_{\mathcal{I}^+}$

$N_{\alpha\beta}$ is unchanged under $\Omega = \ell \rightarrow \omega \ell$, \quad $\omega > 0$

News tensor is geometrically defined tensor field on $\mathcal{I}^+$ independent of choice of conformal factor or $u$-foliation
Bondi-Metzner-Sachs (BMS) Group

BMS group is asymptotic symmetry group

Generators \( \xi^\alpha \) satisfy physical space Killing’s equation at \( I^+ \)

\[ \Omega^2 \mathcal{L}_\xi g^{\alpha\beta}|_{I^+} = -2\Omega^2 \nabla^{(\alpha} \xi^{\beta)}|_{I^+} = 0 \quad \mathcal{L}_\xi \text{ is Lie derivative} \]

**Bondi inertial conformal metric, } \Omega = \ell \]

\[ [\hat{\nabla}^{(\alpha} \xi^{\beta)} - \ell^{-1} \hat{g}^{\alpha\beta} \xi^\gamma \partial_\gamma \ell]|_{\ell=0} = 0 \quad \Rightarrow \quad \xi^\gamma \partial_\gamma \ell = 0 \]

\( \xi^\alpha \) is tangent to \( I^+ \) and \( \ell^{-1} \xi^\alpha \partial_\alpha |_{I^+} = \partial_\ell \xi^\ell |_{I^+} \)

\[ \Rightarrow \quad [\hat{g}^{\alpha\gamma} \partial_\gamma \xi^\beta + \hat{g}^{\beta\gamma} \partial_\gamma \xi^\alpha - \xi^\gamma \partial_\gamma \hat{g}^{\alpha\beta} - \hat{g}^{\alpha\beta} \partial_\ell \xi^\ell]|_{I^+} = 0 \quad \text{where} \]

\[ \hat{g}^{\alpha\beta}|_{I^+} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q^{AB} \end{pmatrix} \]

Only \( \hat{g}^{\alpha\beta}|_{I^+} \) enters \( \Rightarrow \) simple to analyze

**Example } \ell A\text{-component} \]

\[ [\hat{g}^{\ell\gamma} \partial_\gamma \xi^A + \hat{g}^A\gamma \partial_\gamma \xi^\ell]|_{I^+} = \partial_u \xi^A |_{I^+} = 0 \quad \Rightarrow \quad \xi^A |_{I^+} = f^A(x^C) \]

**General solution for BMS generators \]

\[ \xi^\alpha \partial_\alpha|_{\ell=0} = [\alpha(x^C) + \frac{u}{2} \delta_B f^B(x^C)] \partial_u + f^A(x^C) \partial_A \]

\( f^A(x^C) \) is a conformal killing vector on unit sphere

\[ \delta(A f^B) - \frac{1}{2} q^{AB} \delta_C f^C = 0 \]
BMS Group Properties

\[ \xi^\alpha \partial_\alpha \bigg|_{\ell=0} = [\alpha(x^C) + \frac{u}{2} \tilde{\delta}_B f^B(x^C)] \partial_u + f^A(x^C) \partial_A \]

Supertranslations \( f^A = 0 \quad u \to u + \alpha(x^C) \)

Conformal transformations of unit sphere \( \alpha = 0 \)

Isomorphic to the orthochronous Lorentz transformations

Supertranslations: infinite dimensional invariant subgroup

Supertranslations consisting of \( l = (0, 1) \) spherical harmonics, \( \alpha = a + a_x \sin \theta \cos \phi + a_y \sin \theta \sin \phi + a_z \cos \theta \), form an invariant 4-dimensional group of time and space translations

\( \Rightarrow \) Unambiguous definition of energy-momentum

Poincare group consisting of Lorentz transformations and space-time translations is not invariant subgroup

There is a supertranslation freedom in Lorentz transformations analogous to translation freedom (change in origin) in special relativity

Only in special cases, such as non-radiative spacetimes, can a preferred Poincare group be singled out
The Supertranslation Ambiguity

A given cross-section $\Sigma^+$ of $I^+$ picks out a preferred rotation subgroup which maps $\Sigma^+$ into itself.

A supertranslation can map any given cross-section into any other cross-section

$\Rightarrow$ Supertranslation ambiguity in rotation subgroup

Without introducing a preferred structure there is no invariant way to extract a preferred rotation group, Lorentz group or Poincare group from BMS group.

Supertranslation Ambiguity in the Radiation Strain

Under transformation $\tilde{u} = u + \alpha(x^A) + O(\ell)$, with $\tilde{x}^A = x^A$, the radiation strain has supertranslation gauge freedom

$$\tilde{\sigma}(u, x^C) = q^A q^B \nabla_A \nabla_B \tilde{u} |_{I^+} = \sigma(u, x^C) + q^A q^B \bar{\sigma}_A \bar{\sigma}_B \alpha(x^C)$$

At a non-radiative time a preferred Poincare group can be singled out by requiring electric (E-mode) component of the shear to satisfy $\tilde{\sigma}_E = 0$

Determines $\alpha$ up to time and space translation freedom

$$q^A q^B \bar{\sigma}_A \bar{\sigma}_B \alpha = \bar{\sigma}^2 \alpha = 0 \Rightarrow \alpha = a + a_x \sin \theta \cos \phi + a_y \sin \theta \sin \phi + a_z \cos \theta$$
Supertranslation Ambiguity
In Angular Momentum

For any cross-section $\Sigma^+$ of $\mathcal{I}^+$ there is geometrically defined surface integral $Q_{\xi}(\Sigma^+)$ for each BMS generator $\xi^\alpha$ (Wald-Zoupas)

Time and space translations lead to unambiguous construction of Bondi energy-momentum 4-vector

AMBIGUITY IN ANGULAR MOMENTUM IN TRANSITION BETWEEN NON-RADIATIVE STATES

Initial state at $u = -\infty$ picks out inertial frame $\sigma(u = -\infty) = 0$ and preferred Poincare group with rotation generators $\xi^\alpha = R_\alpha^\pm$ with associated angular momentum $Q_{R_\pm}$

Initial inertial frame and its preferred rotation generators extend uniquely to all $\mathcal{I}^+$ but $\sigma(u = +\infty) \neq 0$ in general

Final state picks out inertial frame $\tilde{\sigma}(\tilde{u} = +\infty) = 0$ and Poincare group with rotation generators $\xi^\alpha = R_+^\alpha$ with associated angular momentum $Q_{R_+}$

If $\tilde{u}$ and $u$ differ by a supertranslation the rotation generators $R_+^\alpha$ and $R_-^\alpha$ differ and the corresponding angular momentum differ by supermomenta similar to the Poincare relation $\bar{L} \to \bar{L} + \bar{R} \times \bar{P}$ under a translation of origin by $\bar{R}$

Does this give rise to a distinctively general relativistic mechanism for angular momentum loss via supermomenta?